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The Quaternion Fourier Transform of Finite Measure and Its Properties

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Abstract The main focus of the present article is to propose the quaternion Fourier transform of finite measure, which is a slight generalization of the Fourier transform of finite measure in real and complex domains. Further, we investigate its essential properties, which including translation, Parseval's formula, positive definite, partial derivative, and measure convergence. It is shown that according to the non-commutative nature of quaternion multiplications some properties of the Fourier transform of finite measure are not valid in the quaternion Fourier transform of finite measure.

Keywords Quaternion fourier transform · Finite measure · Bochner's theorem · Parseval's formula

1 Introduction

It is well known that the quaternion Fourier transform [1, 2] is non-trivial extension of the classical Fourier transform in the framework of quaternion algebra. The quaternion Fourier transform is important mathematical tools in areas like applied mathematics and quaternion-valued signal analysis. Basically, the quaternion Fourier transform sends a suitable function on $\mathbb{R}^2$ into a quaternion-valued function. The quaternion Fourier transform of finite measure is intimately related to the quaternion Fourier transform and can be considered as a slight generalization of the quater-

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nion Fourier transform. It transforms a finite measure on $\mathbb{R}^2$ into a quaternion finite measure. The notation of finite measure in the framework of the quaternion Fourier transform and Clifford Fourier transform was introduced by the authors in [3, 4, 15], respectively. They in detail have investigated its interesting properties including the Bochner-Minlos theorem as the fundamental results related to this transformation.

In current article, we aim to continue the work of ones done in [3, 4]. We further investigate useful properties, which were missed such as translation property, Parseval's formula, convergence of measure, partial derivative, and so on. Our first step is to recall the quaternion Fourier transform and its inversion formula, which is needed to define the quaternion Fourier transform of finite measures.

The paper is arranged as follows. In Sect. 2 we shortly remind basic knowledge about quaternion algebra and basic properties that will be needed in later section. Section 3 is concerned with presenting the definition of the quaternion Fourier transform and its inverse. We also discuss σ-bandlimited quaternion function in the quaternion Fourier transform sense. Some of its basic properties are demonstrated in some detail. Section 4 introduce the definition of the quaternion Fourier transform of finite measure. Several properties for the quaternion Fourier transform of finite measure are also discussed in this section. In Sect. 5 we conclude this work.

2 Quaternion Algebra

In this part, we remind the quaternion algebra $\mathbb{H}$ over the real field $\mathbb{R}$, which its elements have the following form

$$\mathbb{H} = \{r = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3 \mid r_0, r_1, r_2, r_3 \in \mathbb{R}\}, \tag{1}$$

where the multiplication law is defined by relation

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2}$$

It is easy to see that from relation (2) the quaternion multiplication is not commutative. The element $r = r_0 + \mathbf{i}\, r_1 + \mathbf{j}\, r_2 + \mathbf{k}\, r_3 \in \mathbb{H}$, r_0 is so-called the *scalar* part of r denoted by $S(r)$ and $\mathbf{i}\, r_1 + \mathbf{j}\, r_2 + \mathbf{k}\, r_3$ is called the *vector* (or *pure*) part of r and denoted by $\mathbf{r}$.

We recall the conjugate $\bar{r}$ of the quaternion r as

$$\bar{r} = r_0 - \mathbf{i}\, r_1 - \mathbf{j}\, r_2 - \mathbf{k}\, r_3. \tag{3}$$

and

$$\bar{r} + r = 2S(r). \tag{4}$$

Applying (3) will lead to the norm or modulus of $r \in \mathbb{H}$ expressed as

$$|r| = \sqrt{r\bar{r}} = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}. \tag{5}$$

Recall that for any $r, p \in \mathbb{H}$. We have

$$\overline{rp} = \bar{p}\bar{r}, \quad |r + p| \leq |r| + |p|. \tag{6}$$

It is straightforward to observe that

$$S(r) \leq |r| \quad \text{and} \quad |\mathbf{r}| \leq |r|. \tag{7}$$

Based on (3) and (5), we can define the inverse of $r \in \mathbb{H} \setminus \{0\}$ as

$$r^{-1} = \frac{\bar{r}}{r\bar{r}}.$$

3 Quaternion Fourier Transform and Bandlimited Functions

Let us start by recalling the definition of the quaternion Fourier transform and its inverse, which will have use of later. A detailed exposition of its properties and application, the reader may consult to [7–14, 16]. We also study σ-bandlimited quaternion function related to the quaternion Fourier transform and its basic properties.

Definition 1 For every $f \in L^2(\mathbb{R}^2; \mathbb{H})$, the quaternion Fourier transform is defined by

$$\mathcal{F}_H\{f\}u, v) = \int_{\mathbb{R}^2} f(x, y)\mathrm{e}^{-\mathbf{i}2\pi ux}\mathrm{e}^{-\mathbf{j}2\pi vy}\, dxdy. \tag{8}$$

The quaternion Fourier transform is invertible, which means that the original function $f(x, y)$ can be expressed in terms $\mathcal{F}_H\{f\}(u, v)$ as shown in the following theorem:

Theorem 1 *The inverse quaternion Fourier transform is given by*

$$f(x, y) = \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u, v)e^{\mathbf{j}2\pi vy}e^{\mathbf{i}2\pi ux}\, du\, dv, \tag{9}$$

provided the integral exists and it is finite.

A useful property of the the quaternion Fourier transform is Plancherel's formula given by

$$\int_{\mathbb{R}^2} |f(x, y)|^2\, dx\, dy = \int_{\mathbb{R}^2} |\mathcal{F}_H\{f\}(u, v)|^2\, du\, dv. \tag{10}$$

Definition 2 We say that a quaternion function $f(x, y)$ is σ-bandlimited if its quaternion Fourier transform can be expressed as

$$\mathcal{F}_H\{f\}(u, v) = 0 \quad for\ u > \sigma, \quad v > \sigma, \tag{11}$$

and the quaternion function $f(x, y)$ is τ-timelimited if it satisfies (compare to [17, 18])

$$f(x, y) = 0 \quad for\ x > \tau, \quad y > \tau. \tag{12}$$

It follows from the inversion theorem for the quaternion Fourier transform (9) that relation (11) may be expressed as

$$f(x, y) = \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \mathcal{F}_H\{f\}(u, v)\mathrm{e}^{\mathbf{j}2\pi vy}\mathrm{e}^{\mathbf{i}2\pi ux}\, du\, dv. \tag{13}$$

Applying Eq. (10) we obtain energy E as

$$E = \int_{\mathbb{R}^2} |f(x, y)|^2\, dx\, dy = \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} |\mathcal{F}_H\{f\}(u, v)|^2\, du\, dv < \infty. \tag{14}$$

The above definition will lead to the following important results.

Theorem 2 *The quaternion Fourier transform of σ-bandlimited function is absolutely integral*

$$\int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} |\mathcal{F}_H\{f\}(u, v)|\, du\, dv < \infty. \tag{15}$$

Proof Applying Schwarz'inequality and Eq. (14) results in

$$\begin{aligned}\left|\int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \left|\mathcal{F}_H\{f\}(u, v)\right| du\, dv\right|^2 &\leq 4\sigma \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \left|\mathcal{F}_H\{f\}(u, v)\right|^2 du\, dv \\ &= 4\sigma E, \end{aligned} \tag{16}$$

and the proof is complete because $E < \infty$.

Theorem 3 *For all $x, y \in \mathbb{R}$, one has*

$$|f(x, y)| \leq 2e^{2\pi\sigma|x|}e^{2\pi\sigma|y|}\sqrt{\sigma E}. \tag{17}$$

Proof If $|u|, |v| < \sigma$, then $\left|\mathrm{e}^{\mathbf{i}2\pi ux}\right| < \mathrm{e}^{2\pi\sigma|x|}$, $\left|\mathrm{e}^{\mathbf{j}2\pi vy}\right| < \mathrm{e}^{2\pi\sigma|y|}$. Furthermore,

$$
\begin{aligned}
|f(x,y)| &= \left| \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \mathcal{F}_H\{f\}(u,v) e^{\mathbf{j}2\pi vy} e^{\mathbf{i}2\pi ux}\, du\, dv \right| \\
&\leq \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \left| \mathcal{F}_H\{f\}(u,v) e^{\mathbf{j}2\pi vy} e^{\mathbf{i}2\pi ux} \right| du\, dv \\
&\leq e^{2\pi\sigma|x|} e^{2\pi\sigma|y|} \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \left| \mathcal{F}_H\{f\}(u,v) \right| du\, dv \\
&\leq 2e^{2\pi\sigma|x|} e^{2\pi\sigma|y|} \sqrt{\sigma E},
\end{aligned}
$$

where in the last equality we have applied relation (16). This finishes the proof of the theorem.

4 Quaternion Fourier Transform of Finite Measure

Before we get into the details of definition of the quaternion Fourier transform of finite measure and its essential properties. We first denote the positive measures on $\mathbb{R}^2$.

$$
M^+(\mathbb{R}^2) = \left\{ \mu | \mu \text{ is a positive measure on } \mathbb{R}^2;\ \mu(\mathbb{R}^2) < \infty \right\}. \tag{18}
$$

This will lead to the following definition.

Definition 3 Given μ in $M^+(\mathbb{R}^2)$. The quaternion Fourier transform of μ is the quaternion function $\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu) : \mathbb{R}^2 \longrightarrow \mathbb{H}$ given by the expression

$$
\begin{aligned}
\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi ux} e^{-\mathbf{j}2\pi vy}\, d\mu(x,y) \\
&= \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi ux} e^{-\mathbf{j}2\pi vy} \mu(x,y)\, dx\, dy.
\end{aligned} \tag{19}
$$

Or

$$
\begin{aligned}
\mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)(u,v) &= \int_{\mathbb{R}^2} e^{-\mathbf{j}2\pi vy} e^{-\mathbf{i}2\pi ux}\, d\mu(x,y) \\
&= \int_{\mathbb{R}^2} e^{-\mathbf{j}2\pi vy} e^{-\mathbf{i}2\pi ux} \mu(x,y)\, dx\, dy.
\end{aligned} \tag{20}
$$

From the above definition, it is straightforward to verify that

$$\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(0,0) = \mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)(0,0) = \mu(\mathbb{R}^2). \tag{21}$$

Further, μ is called a probability measure if and only if $\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(0,0) = 1$.

Example 1 The quaternion Fourier transform of the uniform measure μ on the rectangle $[a,b] \times [c,d]$ is computed as

$$\begin{aligned}\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v) &= \int_a^b \int_c^d \mathrm{e}^{-\mathbf{i}2\pi ux} \mathrm{e}^{-\mathbf{j}2\pi vy}\, dx\, dy \\ &= \frac{\mathrm{e}^{-\mathbf{i}2\pi ub} - \mathrm{e}^{-\mathbf{i}2\pi ua}}{\mathbf{i}2\pi u} \frac{\mathrm{e}^{-\mathbf{j}2\pi vd} - \mathrm{e}^{-\mathbf{j}2\pi vc}}{\mathbf{j}2\pi v}, \quad u \neq 0, v \neq 0. \end{aligned} \tag{22}$$

In what follows, we show that $\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)$ possesses the following properties.

Theorem 4 *The quaternion Fourier transform of $\mu \in M^+(\mathbb{R}^2)$ is bounded and uniformly continuous.*

Proof From the definition, we have

$$\begin{aligned}\left|\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)\right| &\leq \int_{\mathbb{R}^2} \left|\mathrm{e}^{-\mathbf{i}2\pi ux} \mathrm{e}^{-\mathbf{j}2\pi vy}\right| d\mu(x,y) \\ &= \int_{\mathbb{R}^2} d\mu(x,y) \\ &= \mu(\mathbb{R}^2) < \infty, \end{aligned} \tag{23}$$

where the last term of (23) does not depend on u and v and is finite. Now with the aid of (6) we obtain for all $(h_1, h_1) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$$\begin{aligned}&\left|\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u+h_1, v+h_2) - \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)\right| \\ &\leq \int_{\mathbb{R}^2} \left|\mathrm{e}^{-\mathbf{i}2\pi(u+h_1)x} \mathrm{e}^{-\mathbf{j}2\pi(v+h_2)y} - \mathrm{e}^{-\mathbf{i}2\pi ux} \mathrm{e}^{-\mathbf{j}2\pi vy}\right| d\mu(x,y) \\ &= \int_{\mathbb{R}^2} \left|\mathrm{e}^{-\mathbf{i}2\pi ux} \left(\mathrm{e}^{-\mathbf{i}2\pi uh_1} \mathrm{e}^{-\mathbf{i}2\pi vh_2} - 1\right) \mathrm{e}^{-\mathbf{j}2\pi vy}\right| d\mu(x,y) \\ &= \int_{\mathbb{R}^2} \left|\left(\mathrm{e}^{-\mathbf{i}2\pi uh_1} \mathrm{e}^{-\mathbf{i}2\pi vh_2} - 1\right)\right| d\mu(x,y) \\ &\leq \int_{\mathbb{R}^2} \left|\mathrm{e}^{-\mathbf{i}2\pi uh_1} \mathrm{e}^{-\mathbf{i}2\pi vh_2}\right| + |-1|\, d\mu(x,y) \\ &= 2 \int_{\mathbb{R}^2} d\mu(x,y). \end{aligned} \tag{24}$$

Notice that the last term of (24) is the independent of u and v. It also tends to 0 as $h_1 \to 0$ and $h_2 \to 0$. According to the dominated convergence, this proves that $\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)$ is uniformly continuous on $\mathbb{R}^2$.

Theorem 5 *Let $\mu \in M^+(\mathbb{R}^2)$. For every $f \in L^2(\mathbb{R}^2; \mathbb{H})$ we have*

$$\int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v)\, d\mu(x,y) = \int_{\mathbb{R}^2} f(x,y)\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)\, dxdy. \tag{25}$$

Proof We see from (8) and the Fubini theorem that

$$\begin{aligned}\int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v)\, d\mu(x,y) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y)\mathrm{e}^{-\mathbf{i}2\pi ux}\mathrm{e}^{-\mathbf{j}2\pi vy}\, dxdy\, d\mu(x,y)\\ &= \int_{\mathbb{R}^2} f(x,y) \int_{\mathbb{R}^2} \mathrm{e}^{-\mathbf{i}2\pi ux}\mathrm{e}^{-\mathbf{j}2\pi vy}\, d\mu(x,y)\, dxdy\\ &= \int_{\mathbb{R}^2} f(x,y)\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)\, dxdy,\end{aligned} \tag{26}$$

and the proof is complete.

Below we obtain another version of Parseval's formula in the context of the quaternion Fourier transform of μ in the following sense:

Theorem 6 *Let $\mu \in M^+(\mathbb{R}^2)$. Suppose that both f and $\mathcal{F}_H\{f\}$ belong to $L^2(\mathbb{R}^2; \mathbb{H})$. Then we have*

$$\int_{\mathbb{R}^2} f(x,y)\, d\mu(x,y) = \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v)\mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)(-u,-v))\, dudv. \tag{27}$$

Proof By (9) and the Fubini theorem we have

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,y)\, d\mu(x,y) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v)\mathrm{e}^{\mathbf{j}2\pi vy}\mathrm{e}^{\mathbf{i}2\pi ux}\, dudv\, d\mu(x,y)\\ &= \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v)\, dudv \int_{\mathbb{R}^2} \mathrm{e}^{\mathbf{j}2\pi vy}\mathrm{e}^{\mathbf{i}2\pi ux}\, d\mu(x,y)\\ &= \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v)\mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)(-u,-v))\, dudv.\end{aligned} \tag{28}$$

This proves the theorem.

Theorem 7 *Let $\mu \in M^{+}(\mathbb{R}^2)$. For every $b_i \in \mathbb{R}$ with $i = 1, 2$ the following satisfies:*

$$\mathcal{F}_{\mathbf{i},\mathbf{j}}\big(\mu(x - b_1, y - b_2)\big)(u, v) = e^{-\mathbf{i}2\pi u b_1}\mathcal{F}_{\mathbf{i},\mathbf{j}}\big(\mu(x, y)\big)(u, v)e^{-\mathbf{i}2\pi v b_2}. \tag{29}$$

Proof A straightforward computation shows that

$$\begin{aligned}
\mathcal{F}_{\mathbf{i},\mathbf{j}}\big(\mu(x - b_1, y - b_2)\big)(u, v) &= \int_{\mathbb{R}^2} \mathrm{e}^{-\mathbf{i}2\pi u x}\mathrm{e}^{-\mathbf{j}2\pi v y}\, d\mu(x - b_1, y - b_2) \\
&= \int_{\mathbb{R}^2} \mathrm{e}^{-\mathbf{i}2\pi u(b_1+z_1)}\mathrm{e}^{-\mathbf{j}2\pi v(b_2+z_2)}\, d\mu(z_1, z_2) \\
&= \mathrm{e}^{-\mathbf{i}2\pi u b_1}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathbf{i}2\pi u z_1}\mathrm{e}^{-\mathbf{j}2\pi v z_2}\, d\mu(z_1, z_2)\mathrm{e}^{-\mathbf{i}2\pi v b_2} \\
&= \mathrm{e}^{-\mathbf{i}2\pi u b_1}\mathcal{F}_{\mathbf{i},\mathbf{j}}\big(\mu(x, y)\big)(u, v), \mathrm{e}^{-\mathbf{i}2\pi v b_2},
\end{aligned} \tag{30}$$

which was to be proved.

Let us introduce the notion of a positive definite function in the context of quaternion algebra.

Definition 4 A continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{H}$ is said to be positive definite if the inequality

$$\sum_{j,k=1}^{n} b_j \bar{b}_k f(w_j - w_k) \geq 0, \tag{31}$$

is satisfied for any $b_1, b_2, \ldots, b_n \in \mathbb{H}$ and any $w_1, w_2, \ldots, w_n \in \mathbb{R}^2$ with $w_i = (u_i, v_i)$.

It is easily seen that the left-hand side of (31) is a real-valued function. As an immediate consequence of the above definition, we obtain the basic properties of positive definite functions:

Lemma 1 *Let $f : \mathbb{R}^2 \to \mathbb{H}$ be positive definite. Then*

1. $f(0) \geq 0$
2. $f(-u, -v) = \overline{f(u, v)}$ *for all* $(u, v) \in \mathbb{R}^2$.

Proof Taking $n = 1$ and $w_i = 0$ we obtain

$$|b|^2 f(0, 0) \geq 0 \text{ for all } b \in \mathbb{H}. \tag{32}$$

This implies that

$$f(0, 0) \geq 0, \tag{33}$$

which proves item 1. Now taking $n = 2$ and $(u_2, v_2) = (0, 0)$ we get

$$b_1\bar{b}_2 f(u_1, v_1) + b_2\bar{b}_1 f(-u_1, -v_1) \geq 0, \tag{34}$$

which is valid if $f(u_1, v_1) = \overline{f(-u_1, -v_1)}$. This proves item 2.

The next result shows that if μ is a finite positive measure on $\mathbb{R}^2$, then $\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)$ is positive definite.

Theorem 8 *Suppose that $\mu \in M^+(\mathbb{R}^2)$. We have*

$$\sum_{j,k=1}^{n} b_j\bar{b}_k \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(w_j - w_k) \geq 0. \tag{35}$$

Proof By Theorem 4 we have $\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)$ is continuous function. So, we see from the basic properties of quaternion (5) that

$$\begin{aligned}
\sum_{j,k=1}^{n} b_j\bar{b}_k \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(w_j - w_k) &= \int_{\mathbb{R}^2} \sum_{j,k=1}^{n} b_j\bar{b}_k e^{-\mathbf{i}2\pi(u_j-u_k)x} e^{-\mathbf{j}2\pi(v_j-v_k)y}\, d\mu(x, y) \\
&= \int_{\mathbb{R}^2} \sum_{j,k=1}^{n} b_j\bar{b}_k e^{-\mathbf{i}2\pi u_j x} e^{\mathbf{i}2\pi u_k x} e^{-\mathbf{j}2\pi v_j y} e^{\mathbf{j}2\pi v_k y}\, d\mu(x, y) \\
&= \int_{\mathbb{R}^2} \sum_{j,k=1}^{n} b_j\bar{b}_k \overline{e^{\mathbf{i}2\pi u_j x}} e^{\mathbf{i}2\pi u_k x} \overline{e^{\mathbf{j}2\pi v_j y}} e^{\mathbf{j}2\pi v_k y}\, d\mu(x, y) \\
&= \int_{\mathbb{R}^2} \left| \sum_{j=k}^{n} b_j e^{\mathbf{i}2\pi u_j x} e^{\mathbf{j}2\pi v_j y} \right|^2 d\mu(x, y) \\
&\geq 0,
\end{aligned} \tag{36}$$

as desired.

The following definition introduces weak convergence of measures.

Definition 5 Let $\mu, \mu_{i,j} \in M^+(\mathbb{R}^2)$. For every $f \in C_0(\mathbb{R}^2; \mathbb{H})$ we say that $\mu_{i,j}$ converges weakly to μ if

$$\lim_{i,j\to\infty} \int_{\mathbb{R}^2} f(x, y)\, d\mu_{i,j}(x, y) = \int_{\mathbb{R}^2} f(x, y)\, d\mu(x, y), \tag{37}$$

where $C_0(\mathbb{R}^2; \mathbb{H})$ is the space of bounded quaternion continuous functions.

The next result gives a sufficient condition of weak convergence in terms of the quaternion Fourier transform of finite measure.

Theorem 9 *Let μ and $\mu_{i,j}$ in $M^+(\mathbb{R}^2)$. If for each $u, v \in \mathbb{R}^2$,*

$$\lim_{i,j\to\infty} \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu_{i,j})(u, v) = \lim_{i,j\to\infty} \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u, v) \tag{38}$$

Then

$$\lim_{i,j\to\infty} \int_{\mathbb{R}^2} f(x, y)\, d\mu_{i,j}(x, y) = \int_{\mathbb{R}^2} f(x, y)\, d\mu(x, y), \tag{39}$$

for all $f \in L^2(\mathbb{R}^2; \mathbb{H})$.

Proof In fact, from hypothesis we have

$$\mu_{i,j}(\mathbb{R}^2) = \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu_{i,j})(0, 0) \to \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(0, 0). \tag{40}$$

It means that the sequence $\mu_{i,j}(\mathbb{R}^2)$ is bounded. Thus

$$\sup_{i,j} \left|\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu_{i,j})(u, v)\right| = \sup_{i,j} \mu_{i,j}(\mathbb{R}^2) \le C < \infty, \tag{41}$$

for some positive constant C. Further, for every $f \in L^2(\mathbb{R}^2; \mathbb{H})$ we have

$$\begin{aligned} \left|f(u, v)\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu_{i,j})(u, v)\right| &\le \sup_{i,j} \left|f(u, v)\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu_{i,j})(u, v)\right| \\ &\le C|f(u, v)|. \end{aligned} \tag{42}$$

Applying the dominated convergence theorem yields

$$\lim_{i,j\to\infty} \int_{\mathbb{R}^2} f(u, v)\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu_{i,j})(u, v)\, du\, dv = \int_{\mathbb{R}^2} f(u, v)\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u, v)\, du\, dv. \tag{43}$$

Equation (43) can be rewritten as

$$\begin{aligned} \lim_{i,j\to\infty} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} & f(u, v)\mathrm{e}^{-\mathbf{i}2\pi ux}\mathrm{e}^{-\mathbf{j}2\pi vy} du\, dv\, d\mu_{i,j}(x, y) \\ &= \int_{\mathbb{R}^2} f(u, v)\mathrm{e}^{-\mathbf{i}2\pi ux}\mathrm{e}^{-\mathbf{j}2\pi vy}\, du\, dv\, d\mu(x, y). \end{aligned} \tag{44}$$

Or, equivalently,

$$\lim_{i,j\to\infty} \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(x, y)\, d\mu_{i,j}(x, y) = \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(x, y)\, d\mu(x, y). \tag{45}$$

This is the desired result.

Theorem 10 *Let $\mu \in M^+(\mathbb{R}^2)$. If we assume that*

$$\int_{\mathbb{R}^2} |x|\, d\mu(x,y) < \infty, \quad \int_{\mathbb{R}^2} |y|\, d\mu(x,y) < \infty. \tag{46}$$

One has

$$\frac{\partial \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)}{\partial u} = (-\mathbf{i}2\pi) \int_{\mathbb{R}^2} x\, e^{-\mathbf{i}2\pi ux} e^{-\mathbf{j}2\pi vy}\, d\mu(x,y). \tag{47}$$

and

$$\frac{\partial \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)}{\partial v} = \int_{\mathbb{R}^2} y\, e^{-\mathbf{i}2\pi ux} e^{-\mathbf{j}2\pi vy}\, d\mu(x,y)(-\mathbf{j}2\pi). \tag{48}$$

Proof We only prove the first statement. In fact, we have

$$\begin{aligned} &\frac{\mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u+h_1,v) - \mathcal{F}_{\mathbf{i},\mathbf{j}}(\mu)(u,v)}{h_1} \\ &= \int_{\mathbb{R}^2} \frac{1}{h_1}\left(\mathrm{e}^{-\mathbf{i}2\pi(u+h_1)x}\mathrm{e}^{-\mathbf{j}2\pi vy} - \mathrm{e}^{-\mathbf{i}2\pi ux}\mathrm{e}^{-\mathbf{j}2\pi vy}\right) d\mu(x,y) \\ &= \int_{\mathbb{R}^2} \mathrm{e}^{-\mathbf{i}2\pi ux}\frac{\left(\mathrm{e}^{-\mathbf{i}2\pi h_1 x} - 1\right)}{h_1}\mathrm{e}^{-\mathbf{j}2\pi vy}\, d\mu(x,y). \end{aligned} \tag{49}$$

It follows that

$$\begin{aligned} \left|\mathrm{e}^{-\mathbf{i}2\pi ux}\frac{\left(\mathrm{e}^{-\mathbf{i}2\pi h_1 x} - 1\right)}{h_1}\mathrm{e}^{-\mathbf{j}2\pi vy}\right| &= \left|\frac{\mathrm{e}^{-\mathbf{i}2\pi h_1 x} - 1}{h_1}\right| \\ &= \left|\mathrm{e}^{-\mathbf{i}\pi h_1 x}\frac{\left(\mathrm{e}^{-\mathbf{i}\pi h_1 x} - \mathrm{e}^{\mathbf{i}\pi h_1 x}\right)}{h_1}\right| \\ &= 2\left|\frac{\sin \pi h_1 x}{h_1}\right| \\ &\le 2\pi|x|. \end{aligned} \tag{50}$$

Applying the dominated convergence theorem we see that

$$h_1 \to 0, \quad \frac{\mathrm{e}^{-\mathbf{i}2\pi h_1 x} - 1}{h_1} \longrightarrow -\mathbf{i}2\pi x, \tag{51}$$

which yields the assertion.

Theorem 11 *If g is a quaternion continuous function defined on $\mathbb{R}^2$ then it is the quaternion Fourier transform of a positive measure if and only if, there exists a constant C such that*

$$\left| \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v) g(-u,-v))\, dudv \right| \le C \sup_{(x,y)\in\mathbb{R}^2} |f(x,y)|, \tag{52}$$

for every continuous function $f \in L^1(\mathbb{R}^2; \mathbb{H})$.

Proof Taking $g = \mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)$ applying Theorem (6) we see that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v) g(-u,-v))\, dudv \right. &= \left| \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v) \mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)(-u,-v))\, dudv \right| \\ &= \left| \int_{\mathbb{R}^2} f(x,y)\, d\mu(x,y) \right| \\ &\le \sup_{(x,y)} \int_{\mathbb{R}^2} \big| f(x,y) \big|\, d\mu(x,y). \end{aligned} \tag{53}$$

On the other hand, suppose the inequality is valid. The the linear functional g which map f to is a bounded, continuous linear functional on the set of continuous function f. By the Riesz representation theorem, g maps f. Using Parseval's theorem yields

$$\int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v) g(-u,-v))\, dudv = \int_{\mathbb{R}^2} \mathcal{F}_H\{f\}(u,v) \mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)(-u,-v))\, dudv. \tag{54}$$

This must hold for all f. So, $g = \mathcal{F}_{\mathbf{j},\mathbf{i}}(\mu)$.

5 Conclusion

The Fourier transform of finite measure has been extended to the quaternion plane so-called the quaternion Fourier transform of finite measure. Several essential properties including translation, Parseval's formula, positive definite, partial derivative, and measure convergence are studied. These properties are generalizations of corresponding properties of the Fourier transform of finite measure in real and complex regions. We have shown that some properties of the Fourier transform of finite measure does not hold in the setting of the quaternion Fourier transform of finite measure.

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